Ex $x^2p + xyq = y^2$.

Comparing with Lagrange's equation $Pp + Qq = R$ we have Lagrange's auxiliary equation as

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dz}{y^2} \quad -(1)$$

Hence from (1)

$$\frac{dx}{x^2} = \frac{dy}{xy}$$

$$\Rightarrow \cancel{\frac{dx}{x^2}} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y} + \ln c_1$$

$$\Rightarrow \ln x = \ln y + \ln c_1$$

$$\Rightarrow \frac{x}{y} = c_1$$

c_1 = constant of integration.

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From (1) we can also write

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dz}{y^2} \quad \cancel{= \frac{dx - dy}{x^2 - xy}}$$

$$= \frac{-\frac{y^2}{x^2}dx + \frac{2y}{x}dy}{\left(-\frac{y^2}{x^2}\right)x^2 + \frac{2y}{x}xy}$$

$$\left[\because \frac{p}{q} = \frac{r}{s} = \frac{\alpha p + \beta r}{\alpha q + \beta s}\right]$$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{xy} = \frac{dz}{y^2} = \frac{-\frac{y^2}{x^2}dx + \frac{2y}{x}dy}{y^2}$$

So $$\frac{dz}{y^2} = \frac{-\frac{y^2}{x^2}dx + \frac{2y}{x}dy}{y^2}$$

$$\Rightarrow dz = -\frac{y^2}{x^2}dx + \frac{2y}{x}dy = d\left(\frac{y^2}{x}\right)$$

$$\int dz = \int d\left(\frac{y^2}{x}\right)$$

$$\Rightarrow z = \frac{y^2}{x} + c_2 \quad [c_2 = \text{constant of integration}]$$

$$\phi_1 \Rightarrow z - \frac{y^2}{x} = c_2$$

Hence General solution is $\phi\left(z - \frac{y^2}{x}, \frac{x}{y}\right) = 0$.

Here ϕ is an arbitrary function.

Ex $xp + yq = \sqrt{x^2+y^2}$

Comparing with Lagrange's Equation $Pp + Qq = R$ we have Lagrange's Auxiliary Equation as:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{\sqrt{x^2+y^2}} \quad -(1)$$

So from (1)

$$\frac{dx}{x} = \frac{dy}{y}$$

$\Rightarrow \ln x = \ln y + \ln c_1$ [by integration]

$\Rightarrow \frac{x}{y} = c_1$ [c_1 = constant of integration.

Now from (1)

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{\sqrt{x^2+y^2}} = \frac{xdy + ydy}{x^2+y^2}$$

$$\left[\because \frac{dx}{x} = \frac{dy}{y} = \frac{xdx + ydy}{x^2+y^2}.\right]$$

So $\frac{dz}{\sqrt{x^2+y^2}} = \frac{xdy + ydy}{x^2+y^2}$

$\Rightarrow dz = \frac{xdy + ydy}{\sqrt{x^2+y^2}} = \frac{1}{2} \frac{d(x^2+y^2)}{\sqrt{x^2+y^2}}$

$[\because d(x^2+y^2) = d(x^2) + d(y^2)$
$= 2xdx + 2ydy.]$

$\Rightarrow \int dz = \frac{1}{2}\int \frac{d(x^2+y^2)}{\sqrt{x^2+y^2}} + c_2$ [Integrating]

$$\Rightarrow z = \frac{1}{2}\int \frac{du}{\sqrt{u}} + c_2 \qquad [c_2 = \text{constant of integration}]$$

$$[\cancel{u = \sqrt{r}} \quad u = x^2 + y^2]$$

$$\Rightarrow \cancel{z = \frac{1}{2}\cdot u} \quad \Rightarrow z = \sqrt{u} + c_2$$

$$\Rightarrow z = \sqrt{x^2 + y^2} + c_2$$

$$\Rightarrow z - \sqrt{x^2 + y^2} = c_2$$

Hence the general solution is

$$\phi\left(z - \sqrt{x^2 + y^2}, \frac{x}{y}\right) = 0$$

where ϕ is an arbitrary ~~constant~~ function.